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A Kind of Four-Mode Nonlinear Entangled State Representation in Quantum Optics

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Abstract Based on the technique of integration within an ordered product of nonlinear bosonic operators, we construct a new four-mode nonlinear entangled state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ in 4-mode Fock space, which can make up a complete set. Its properties and applications are discussed. A possible scheme to generate this state is also presented.

Keywords IWOP technique · Nonlinear entangled state

1 Introduction

The conception of quantum entanglement has become more and more fascinating and important, since the publication of Einstein-Podolsky-Rosen' (EPR) paper in 1935 [1], arguing the incompleteness of quantum mechanics. In recent years quantum entanglement plays a central role in quantum information processing [2–4]. Some entangled states have been introduced, such as bipartite entangled states [5, 6], tripartite entangled states [7, 8] and four-mode entangled states [9]. On the other hand, nonlinear coherent states (NCS) [10–20] have been paid much attention because they can be realized as the stationary states of center-of-mass motion of a trapped ion. Squeezed states and phase states, etc., can also be viewed as some nonlinear coherent states. The NCS $|z\rangle_f$ is defined as an eigenstate of $f(N)a$, where $f(N)$ is an operator-valued function of the number operator, $N=a^\dagger a$, $[a,a^\dagger]=1$,

$$f(N)\,a\,|z\rangle_f = z\,|z\rangle_f\,. \tag{1}$$

Due to $\frac{1}{f(N-1)}a^\dagger = a^\dagger\frac{1}{f(N)}$, one can see

$$\left[f(N)a,\frac{1}{f(N-1)}a^\dagger\right]=1, \tag{2}$$

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and

$$|z\rangle_f = \exp\left[\frac{z}{f(N-1)}a^\dagger\right]|0\rangle . \tag{3}$$

The bra is given by [6]

$$_f\langle\langle z| = \left(|z\rangle\rangle_f\right)^\dagger = \langle 0|\exp\left[z^* f(N)a\right], \qquad |z\rangle\rangle_f = \exp\left[za^\dagger f(N)\right]|0\rangle . \tag{4}$$

Based on the normal ordering form of vacuum projector [21]

$$|0\rangle\langle 0| = {}^{\circ}_{\circ}\exp\left[-\frac{1}{f(N-1)}a^\dagger f(N)a\right]{}^{\circ}_{\circ}, \tag{5}$$

where the symbol ${}^{\circ}_{\circ}\,{}^{\circ}_{\circ}$ means normal ordering of $\frac{1}{f(N-1)}a^\dagger$ standing on the left of $f(N)a$, and using the technique of integral within an ordered product (IWOP) of nonlinear [22, 23] Bose operators $f(N)a$ and $\frac{1}{f(N-1)}a^\dagger$ as well as the following integral formula

$$\int \frac{d^2z}{\pi}\exp\left[\xi|z|^2+\zeta z+\eta z^*\right] = -\frac{1}{\xi}\exp\left[-\frac{\zeta\eta}{\xi}\right], \quad \mathrm{Re}\,\xi<0, \tag{6}$$

the resolution of unity in terms of the NCS is shown as

$$\int \frac{d^2z}{\pi}e^{-|z|^2}\,|z\rangle_{ff}\,\langle\langle z| = 1. \tag{7}$$

We should emphasize that the bra and the ket in (7) are not mutual Hermite conjugate.

In the past decade quantum entangled states have been studied and employed intensively because of their potential uses in quantum communication and quantum computation [1–4, 24, 25]. Thus it is natural to challenge us to combine the nonlinear Bose operators and the concept of quantum entanglement to construct some nonlinear entangled states representation. However, due to the fact that quantum entanglement involves at least two modes and the complexity arising from the nonlinear property, how to extend the Einstein-Podolsky-Rosen entangled state [6, 26] (the common eigenvector of the operators $a-b^\dagger$ and $a^\dagger-b$),

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2+\eta a^\dagger-\eta^* b^\dagger+a^\dagger b^\dagger\right]|00\rangle , \tag{8}$$

where $[a,a^\dagger]=[b,b^\dagger]=1$, to nonlinear case is not a easy task, because we should determine what is the appropriate nonlinear operators, which includes two-mode number operator-valued function and operators $a-b^\dagger$ and $a^\dagger-b$. After considerable amount of trying works, $f(Q)(a-b^\dagger)$ and $\frac{1}{f(Q-1)}(a^\dagger-b)$ are chosen, $f(Q)$ is an operator-valued function of the number-difference operator $Q=aa^\dagger-bb^\dagger$, and a type of bipartite nonlinear entangle states (NES) $|\eta\rangle_f$ has been construct [27]

$$|\eta\rangle_f = \exp\left[-\frac{1}{2}|\eta|^2+\frac{\eta}{f(Q-1)}a^\dagger-\eta^* f(Q)b^\dagger+a^\dagger b^\dagger\right]|00\rangle , \tag{9}$$

which satisfies the following eigen equations,

$$f(Q)(a-b^\dagger)\,|\eta\rangle_f = \eta\,|\eta\rangle_f ,$$

$$\frac{1}{f(Q-1)}(a^\dagger - b)\,|\eta\rangle_f = \eta^*\,|\eta\rangle_f\,. \tag{10}$$

In particular when $f(Q) = f(Q-1) = 1$, (9) reduces to the ordinary bipartite entangled state.

From (9), one can see that for constructing the corresponding bipartite NESs, $f(Q)$ is introduced to replace $f(N)$ in (1). Using the IWOP technique [28, 29] the completeness relation can be shown

$$\int \frac{d^2\eta}{\pi}\,|\eta\rangle_{ff}\,\langle\langle\eta| = 1, \tag{11}$$

where

$${}_f\langle\langle\eta| = \langle 00|\exp\left[-\frac{1}{2}|\eta|^2 + \eta^* f(Q)a - \eta\frac{1}{f(Q-1)}b + ab\right], \tag{12}$$

and ${}_f\langle\langle\eta|$ is not the Hermite conjugate of $|\eta\rangle_f$.

Multipartite nonlinear entanglement in pure states of many systems is a founding property and a crucial resource for quantum information science. In this paper, as an extension of the three-mode NES proposed by Fan et al., we shall introduce a new four-mode NES in Fock space, denoted by $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$, which can compose a new quantum mechanical representation. The work is arranged as follows: In Sect. 2 we construct and analyze $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$. In Sect. 3 we design how to implement it theoretically. Section 4 is devoted to deriving its completeness property and partly non-orthogonal property. The Schmidt decomposition and the quadrature amplitude measurement of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ are demonstrated in Sect. 5. In the last section an application of the $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ is simply considered.

2 A New Four-Mode NES

In order to construct a new kind of four-mode NES, we must first determine an operator-valued function $f(\mathbf{Q})$ which involves four-mode number operators. After many tries we choose

$$\mathbf{Q} = (a^\dagger a + d^\dagger d) - (b^\dagger b + c^\dagger c), \tag{13}$$

where $d^\dagger, d$ are the fourth mode. Using $f(\mathbf{Q})$ we define

$$A = \frac{1}{f(\mathbf{Q}-1)}a, \qquad B = f(\mathbf{Q})b, \qquad C = f(\mathbf{Q})c, \qquad D = \frac{1}{f(\mathbf{Q}-1)}d, \tag{14}$$

and

$$A^+ = f(\mathbf{Q})a^\dagger, \qquad B^+ = \frac{1}{f(\mathbf{Q}-1)}b^\dagger, \qquad C^+ = \frac{1}{f(\mathbf{Q}-1)}c^\dagger, \qquad D^+ = f(\mathbf{Q})d^\dagger, \tag{15}$$

with Bose operators obeying $[c, c^\dagger] = 1$, $[d, d^\dagger] = 1$. Note that A^+ is not the Hermitian conjugate of A, i.e., '+' does not mean the Hermite conjugate, $(A^+)^\dagger \neq A$, $(B^+)^\dagger \neq B$, $(C^+)^\dagger \neq C$, $(D^+)^\dagger \neq D$. Noticing

$$\mathbf{Q}a^\dagger = a^\dagger(\mathbf{Q}-1), \qquad \mathbf{Q}b^\dagger = b^\dagger(\mathbf{Q}+1), \tag{16}$$

one can prove

$$[A, A^+] = [B, B^+] = [C, C^+] = [D, D^+] = 1, \tag{17}$$

the commutative relations of other forms are equal zero.

Now let us write down the four-mode NES in Fock space,

$$|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = \operatorname{sech}\lambda\operatorname{sech}\mu\exp\Big[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2+|\gamma|^2\right)+(A^+-\alpha^*)(B^+-\beta^*)\tanh\lambda$$
$$+C^+(D^+-\gamma^*)\tanh\mu+(A^+-\alpha^*)C^+\operatorname{sech}\lambda\operatorname{sech}\mu$$
$$+\alpha A^+ + \beta B^+ + \gamma D^+\Big]|0000\rangle\,, \tag{18}$$

where $|0000\rangle$ is annihilated by a, b, c and d, so it is also annihilated by A, B, C and D, $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$, and $\gamma=\gamma_1+i\gamma_2$ are complex numbers, λ and μ are two real numbers. It is evident that when $\lambda=\mu=0$, (18) becomes

$$|\alpha,\beta,\gamma\rangle_{0,0} = \exp\Big[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2+|\gamma|^2\right)+(A^+-\alpha^*)C^+ + \alpha A^+ + \beta B^+ + \gamma D^+\Big]|0000\rangle$$
$$= |\eta=\alpha\rangle_{1,3}\otimes|\beta\rangle_2\otimes|\gamma\rangle_4\,, \tag{19}$$

where $|\eta=\alpha\rangle_{1,3}$ is the bipartite NES in A–C mode [30–32], $|\beta\rangle_2$ and $|\gamma\rangle_4$ are the NCS in B and D modes, respectively. In addition, (18) will reduce to a tripartite NES with $\gamma=\mu=0$.

Operating A, B, C, and D on $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ we obtain four independent eigenvector equations

$$(D-C^+\tanh\mu)\,|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = \gamma\,|\alpha,\beta,\gamma\rangle_{\lambda,\mu}\,, \tag{20}$$

$$(B-A^+\tanh\lambda)\,|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = \left(\beta-\alpha^*\tanh\lambda\right)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}\,, \tag{21}$$

$$\left(A-B^+\tanh\lambda-C^+\operatorname{sech}\lambda\operatorname{sech}\mu\right)|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = \left(\alpha-\beta^*\tanh\lambda\right)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}\,, \tag{22}$$

$$\left(C-A^+\operatorname{sech}\lambda\operatorname{sech}\mu-D^+\tanh\mu\right)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$$
$$= -\left(\gamma^*\tanh\mu+\alpha^*\operatorname{sech}\lambda\operatorname{sech}\mu\right)|\alpha,\beta,\gamma\rangle_{\lambda,\mu}\,. \tag{23}$$

By introducing $X_1=\frac{1}{\sqrt{2}}(A+A^+)$, $P_1=\frac{1}{\sqrt{2}i}(A-A^+)$, combining (22) and (20) we have

$$[(X_1\cosh\lambda-X_2\sinh\lambda)\operatorname{sech}\mu-X_3+X_4\tanh\mu]\,|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$$
$$=\sqrt{2}\left[(\alpha_1\cosh\lambda-\beta_1\sinh\lambda)\operatorname{sech}\mu+\gamma_1\tanh\mu\right]|\alpha,\beta,\gamma\rangle_{\lambda,\mu}\,, \tag{24}$$

$$[(P_1\cosh\lambda+P_2\sinh\lambda)\operatorname{sech}\mu+P_3+P_4\tanh\mu]\,|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$$
$$=\sqrt{2}\left[(\alpha_2\cosh\lambda+\beta_2\sinh\lambda)\operatorname{sech}\mu+\gamma_2\tanh\mu\right]|\alpha,\beta,\gamma\rangle_{\lambda,\mu}\,. \tag{25}$$

Note that $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ is the common eigenvector of the above four operators $(B-A^+\tanh\lambda)$, $(C-A^+\operatorname{sech}\lambda\operatorname{sech}\mu-D^+\tanh\mu)$, $[(X_1\cosh\lambda-X_2\sinh\lambda)\operatorname{sech}\mu-X_3+X_4\tanh\mu]$, and $[(P_1\cosh\lambda+P_2\sinh\lambda)\operatorname{sech}\mu+P_3+P_4\tanh\mu]$, which constitute a complete commutable operator set.

3 Generation of the NES $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$

Next we will discuss how to produce the ideal four-mode entangled state $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$. For this purpose we introduce two squeezing transformation [33]

$$\begin{aligned} S_{12}A^{+}S_{12}^{-1} &= A^{+}\cosh\lambda - B\sinh\lambda, \\ S_{34}C^{+}S_{34}^{-1} &= C^{+}\cosh\mu - D\sinh\mu, \end{aligned} \tag{26}$$

where S_{12}, S_{34} are two nonlinear two-mode squeezing operators, $S_{12} = \exp[\lambda(A^{+}B^{+} - AB)]$ and $S_{34} = \exp[\mu(C^{+}D^{+} - CD)]$, then we have

$$\begin{aligned} &S_{12}S_{34}\exp[A^{+}C^{+}]\,|0000\rangle \\ &\quad = \operatorname{sech}\lambda\operatorname{sech}\mu\exp[A^{+}C^{+}\cosh\lambda\cosh\mu]\exp[-A^{+}D\cosh\lambda\sinh\mu] \\ &\qquad \times\exp[-BC^{+}\sinh\lambda\cosh\mu]\exp[B\sinh\lambda D\sinh\mu]\exp[A^{+}B^{+}\tanh\lambda] \\ &\qquad \times\exp[C^{+}D^{+}\tanh\mu]\,|0000\rangle\,. \end{aligned} \tag{27}$$

Using the Baker-Hausdorff formula $e^{\rho}e^{\sigma} = e^{\sigma}e^{\rho}e^{[\rho,\sigma]}$, valid for $[\rho,[\rho,\sigma]] = [\sigma,[\rho,\sigma]] = 0$, we obtain

$$\begin{aligned} &\exp[-A^{+}D\cosh\lambda\sinh\mu]\exp[C^{+}D^{+}\tanh\mu] \\ &\quad = \exp[C^{+}D^{+}\tanh\mu]\exp\left[-A^{+}C^{+}\tanh\mu\sinh\mu\cosh\lambda\right]\exp[-A^{+}D\cosh\lambda\sinh\mu]. \end{aligned} \tag{28}$$

Substituting (28) into (27) we see

$$\begin{aligned} &S_{12}S_{34}\exp[A^{+}C^{+}]\,|0000\rangle \\ &\quad = \operatorname{sech}\lambda\operatorname{sech}\mu\exp[A^{+}B^{+}\tanh\lambda + C^{+}D^{+}\tanh\mu + A^{+}C^{+}\operatorname{sech}\lambda\operatorname{sech}\mu]\,|0000\rangle \\ &\quad = |\alpha = 0, \beta = 0, \gamma = 0\rangle_{\lambda,\mu}\,. \end{aligned} \tag{29}$$

Then making a three single-mode nonlinear displacement $D_1(\alpha)D_2(\beta)D_4(\gamma)$ for $|\alpha = 0, \beta = 0, \gamma = 0\rangle_{\lambda,\mu}$, where $D_1(\alpha) = \exp[\alpha A^{+} - \alpha^{*}A]$, we can make up the $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ state

$$D_1(\alpha)D_2(\beta)D_4(\gamma)|\alpha = 0, \beta = 0, \gamma = 0\rangle_{\lambda,\mu} = |\alpha, \beta, \gamma\rangle_{\lambda,\mu}. \tag{30}$$

Thus the new nonlinear 4-mode NES is generated.

4 Properties of $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$

We now investigate the main properties of $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$. Using the normal ordered expansion (in the sense of $\substack{\circ\,\circ\\ \circ\,\circ}$) of $|0000\rangle\langle 0000|$ and letting $:\ :$ denote the usual normal ordering symbol regarding to $a^{\dagger}$ and a, considering the result of (17) we have

$$\begin{aligned} &:(a^{\dagger}a + b^{\dagger}b + c^{\dagger}c + d^{\dagger}d)^{n}: \\ &\quad = \sum_{i+j+k=n}\frac{n!}{i!j!k!l!}a^{\dagger i}b^{\dagger j}c^{\dagger k}d^{\dagger l}a^{i}b^{j}c^{k}d^{l} \end{aligned}$$

$$= \sum_{i+j+k=n} \frac{n!}{i!j!k!l!} [f(\mathbf{Q})a^\dagger]^i \left[\frac{1}{f(\mathbf{Q}-1)} b^\dagger\right]^j \left[\frac{1}{f(\mathbf{Q}-1)} c^\dagger\right]^k [f(\mathbf{Q})d^\dagger]^l$$
$$\times \left(\frac{1}{f(\mathbf{Q}-1)} a\right)^i (f(\mathbf{Q})b)^j (f(\mathbf{Q})c)^k \left(\frac{1}{f(\mathbf{Q}-1)} d\right)^l$$
$$= \sum_{i+j+k=n} \frac{n!}{i!j!k!} A^{+i} B^{+j} C^{+k} D^{+l} A^i B^j C^k D^l$$
$$= {}^{\circ}_{\circ} (A^+A + B^+B + C^+C + D^+D)^n {}^{\circ}_{\circ}. \tag{31}$$

It then follows that

$$|0000\rangle\langle 0000| = :\exp(-a^\dagger a - b^\dagger b - c^\dagger c - d^\dagger d):$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} : (a^\dagger a + b^\dagger b + c^\dagger c + d^\dagger d)^n :$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} {}^{\circ}_{\circ} (A^+A + B^+B + C^+C + D^+D)^n {}^{\circ}_{\circ}, \tag{32}$$

thus we see

$$|0000\rangle\langle 0000| = {}^{\circ}_{\circ} \exp[-A^+A - B^+B - C^+C - D^+D] {}^{\circ}_{\circ}. \tag{33}$$

On the other hand, we introduce the bra

$${}_{\lambda,\mu}\langle\langle \alpha,\beta,\gamma| = \langle 0000| \operatorname{sech}\lambda \operatorname{sech}\mu \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 + |\gamma|^2\right) + (A-\alpha)(B-\beta)\tanh\lambda\right.$$
$$\left. + C(D-\gamma)\tanh\mu + (A-\alpha)C \operatorname{sech}\lambda \operatorname{sech}\mu + \alpha^* A + \beta^* B + \gamma^* D\right], \tag{34}$$

then using the IWOP technique and (6), we can prove the completeness relation

$$\cosh^2\mu \int \frac{d^2\alpha d^2\beta d^2\gamma}{\pi^3} |\alpha,\beta,\gamma\rangle_{\lambda,\mu\lambda,\mu}\langle\langle \alpha,\beta,\gamma|$$
$$= {}^{\circ}_{\circ} \exp\left[\frac{1}{\operatorname{sech}^2\lambda}(B^+ - A\tanh\lambda + E\tanh\lambda)\left(B - A^+\tanh\lambda + E^\dagger\tanh\lambda\right)\right]$$
$$\times \exp[(D^+ - C\tanh\mu)\left(D - C^+\tanh\mu\right) + E^\dagger E + E_1 + E_2 + E_3 + K] {}^{\circ}_{\circ}$$
$$= {}^{\circ}_{\circ} \exp[0] {}^{\circ}_{\circ} = 1, \tag{35}$$

where the factor $\cosh^2\mu$ is integration measurement needed for the completeness relation, and

$$K = -A^+A - B^+B - C^+C - D^+D,$$

$$E_1 = (A^+B^+ + AB)\tanh\lambda, \qquad E^\dagger = A^+ - B\tanh\lambda - C\,\mathrm{sech}\,\lambda\,\mathrm{sech}\,\mu, \tag{36}$$
$$E_2 = \left(C^+D^+ + CD\right)\tanh\mu, \qquad E_3 = \left(A^+C^+ + AC\right)\mathrm{sech}\,\lambda\,\mathrm{sech}\,\mu.$$

In order to find out the explicit form of ${}_{\lambda,\lambda}\langle\langle\alpha',\beta',\gamma'|\alpha,\beta,\gamma\rangle_{\lambda,\lambda}$ (here for simplicity, we consider the case of $\lambda=\mu$), we appeal to the four-mode NCSs

$$|z_1, z_2, z_3, z_4\rangle_f = \exp[z_1A^+ + z_2B^+ + z_3C^+ + z_4D^+]\,|0000\rangle\,, \tag{37}$$

$${}_f\langle\langle z_1, z_2, z_3, z_4| = \langle 0000|\exp[z_1^*A + z_2^*B + z_3^*C + z_4^*D], \tag{38}$$

and calculate

$$\begin{aligned}
&{}_f\langle\langle z_1, z_2, z_3, z_4|\,\alpha,\beta,\gamma\rangle_{\lambda,\mu} \\
&\quad= \mathrm{sech}\,\lambda\,\mathrm{sech}\,\mu\exp\left[-\frac{1}{2}\left(\sum_{i=1}^{4}|z_i|^2 + |\alpha|^2 + |\beta|^2 + |\gamma|^2\right) + (z_1^*-\alpha^*)(z_2^*-\beta^*)\tanh\lambda\right] \\
&\qquad\times\exp[z_3^*(z_4^*-\gamma^*)\tanh\mu + (z_1^*-\alpha^*)z_3^*\mathrm{sech}\,\lambda\,\mathrm{sech}\,\mu + \alpha z_1^* + \beta z_2^* + \gamma z_4^*].
\end{aligned} \tag{39}$$

Using the overcompleteness relation

$$\int\prod_{i=1}^{4}\left(\frac{d^2z_i}{\pi}\right)\exp\left(-\sum_{i=1}^{4}|z_i|^2\right)|z_1, z_2, z_3, z_4\rangle_{ff}\langle\langle z_1, z_2, z_3, z_4| = 1, \tag{40}$$

we finally obtain

$$\begin{aligned}
&{}_{\lambda,\lambda}\langle\langle\alpha',\beta',\gamma'|\,\alpha,\beta,\gamma\rangle_{\lambda,\lambda} \\
&\quad= GH\mathrm{sech}^4\lambda\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\exp\left[-\frac{\mathrm{sech}^2\lambda}{\varepsilon}|\alpha'-\alpha-(\beta'^*-\beta^*)\tanh\lambda\right. \\
&\qquad\left.+(\gamma'-\gamma)\tanh\lambda|^2\right],
\end{aligned} \tag{41}$$

where

$$G = \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + \left|\alpha'\right|^2 + \left|\beta'\right|^2 + \left|\gamma'\right|^2)\right], \tag{42}$$

and

$$\begin{aligned}
H &= \exp\left[\cosh^2\lambda(\beta'^* + (\alpha-\alpha')\tanh\lambda - \beta^*\tanh^2\lambda)\left(\beta + (\alpha'^*-\alpha^*)\tanh\lambda - \beta'\tanh^2\lambda\right)\right] \\
&\quad\times\exp\left[\coth^2\lambda\left(\alpha-\alpha'-(\beta^*-\beta'^*+\gamma')\tanh\lambda\right)(\alpha'^*-\alpha^*-(\beta'-\beta+\gamma^*)\tanh\lambda)\right] \\
&\quad\times\exp\left[(\alpha-\beta^*\tanh\lambda)\left(\alpha'^*-\beta'\tanh\lambda\right) + \left(\alpha^*\beta^*+\alpha'\beta'\right)\tanh\lambda\right].
\end{aligned} \tag{43}$$

Using the following limiting formula of δ-function:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\exp\left[-\frac{1}{\varepsilon}|\alpha|^2\right] = \pi\delta(\alpha)\,\delta\left(\alpha^*\right) = \pi\delta(\alpha_1)\,\delta(\alpha_2)\,, \quad \alpha = \alpha_1 + i\alpha_2, \tag{44}$$

we obtain

$$
\begin{aligned}
{}_{\lambda,\lambda}\langle\langle \alpha', \beta', \gamma' |\, \alpha, \beta, \gamma\rangle_{\lambda,\lambda} &= \pi G H \operatorname{sech}^2\lambda\, \delta\big(\alpha' - \alpha - \big(\beta'^* - \beta^*\big)\tanh\lambda + (\gamma' - \gamma)\tanh\lambda\big) \\
&\quad \times \delta\left(\alpha'^* - \alpha^* - (\beta' - \beta)\tanh\lambda + (\gamma'^* - \gamma^*)\tanh\lambda\right).
\end{aligned} \tag{45}
$$

From (45) we see that $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ is partly orthonormal. In addition, we should point out that when $\lambda \neq \mu$ $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ state vectors are not orthogonal to each other.

5 Quadrature Amplitude Measurement on $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$

To see the entanglement involved in $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ more clearly, we make the following three-fold Fourier transformation for it. After a long but straightforward calculation we obtain

$$
\int_{-\infty}^{\infty} d\alpha_2 d\beta_2 d\gamma_2\, |\alpha, \beta, \gamma\rangle_{\lambda,\mu}\, e^{iu\alpha_2 + i\upsilon\beta + i\sigma\gamma_2} = W(u, \upsilon, \sigma)\, |x_1\rangle_1 \otimes |x_2\rangle_2 \otimes |x_3\rangle_3 \otimes |x_4\rangle_4 \tag{46}
$$

where the four single-mode states all belong to the nonlinear coordinate eigenvectors,

$$
\begin{aligned}
|x_i\rangle_i &= \pi^{-\frac{1}{4}} \exp\left[-\frac{x_i^2}{2} + \sqrt{2}x_i F^+ - \frac{F^{+2}}{2}\right] |0\rangle_i\,, \\
{}_i\,\langle\langle x_i| &= {}_i\,\langle 0|\, \pi^{-\frac{1}{4}} \exp\left[-\frac{x_i^2}{2} + \sqrt{2}x_i F - \frac{F^2}{2}\right]
\end{aligned} \tag{47}
$$

where $F = A,\ B,\ C,\ D$. $i = 1, 2, 3, 4$, and $W(u, \upsilon, \sigma)$ is a normalization factor (not be written),

$$
\begin{aligned}
x_1 &= \frac{(\alpha_1 - u)}{\sqrt{2}}, \qquad x_2 = \frac{\beta_1 - \upsilon}{\sqrt{2}}, \qquad x_4 = \frac{\gamma_1 - \sigma}{\sqrt{2}}, \\
x_3 &= \frac{1}{\sqrt{2}}\{[(\beta_1 + \upsilon)\sinh\lambda - (\alpha_1 + u)\cosh\lambda]\operatorname{sech}\mu - (\gamma_1 + \sigma)\tanh\mu\}.
\end{aligned} \tag{48}
$$

Then making the inverse transformation of (46) we obtain

$$
|\alpha, \beta, \gamma\rangle_{\lambda,\mu} = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} du d\upsilon d\sigma\, W(u, \upsilon, \sigma) e^{-(iu\alpha_2 + i\upsilon\beta + i\sigma\gamma_2)}\, |x_1\rangle_1 \otimes |x_2\rangle_2 \otimes |x_3\rangle_3 \otimes |x_4\rangle_4\,, \tag{49}
$$

which is just the Schmidt decomposition [34] of $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$, this in turn confirms that $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$ is an entangled state.

When making a single-mode quadrature amplitude measurement, say $|x\rangle_{11}\langle\langle x|$ on $|\alpha, \beta, \gamma\rangle_{\lambda,\mu}$, where $|x\rangle$ is the nonlinear coordinate eigenstate of $X = \frac{F + F^+}{\sqrt{2}}$ operator, where $F = A,\ B,\ C,\ D$. Using (49) we obtain

$$
\begin{aligned}
{}_1\,\langle\langle x \,|\, \alpha, \beta, \gamma\rangle_{\lambda,\mu} &= \sqrt{2} e^{-iu\alpha_2} \frac{1}{8\pi^3} \int_{-\infty}^{\infty} d\upsilon d\sigma\, W(\upsilon, \sigma) e^{-(i\upsilon\beta + i\sigma\gamma_2)} \\
&\quad \times |x_2\rangle_2 \otimes |x_3\rangle_3 \otimes |x_4\rangle_4 \Big|_{u = \alpha_1 - \sqrt{2}x},
\end{aligned} \tag{50}
$$

which shows that the remaining three modes are kept in entanglement. If a joint Bell measurement, say $|\eta'\rangle_{1,21,2}\langle\langle\eta'|$ shown in (9), is performed on $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$, the result of measurement is

$$_{1,2}\langle\langle\eta'|\alpha,\beta,\gamma\rangle_{\lambda,\mu} = e^{M}\frac{\operatorname{sech}\lambda\operatorname{sech}\mu}{1-\tanh\lambda}\exp\left[\tau C^{+}+\gamma D^{+}+C^{+}D^{+}\tanh\mu\right]|00\rangle\,, \tag{51}$$

where e^{M} is an unimportant coefficient and

$$\tau = \frac{\operatorname{sech}\lambda\operatorname{sech}\mu}{1-\tanh\lambda}\left(\eta'^{*}-\alpha^{*}+\beta\right)-\gamma^{*}\tanh\mu. \tag{52}$$

Equation (51) indicates that after the Bell joint measurement the remaining two-mode is still in entanglement.

6 Application of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$

Following Hu et al. [9] we consider an application of the $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ in quantum teleportation. From (35) we know that any 4-mode nonlinear state $|\,\rangle_{1234}$ can be expanded as

$$|\,\rangle_{1234} = \int\frac{d^{2}\alpha d^{2}\beta d^{2}\gamma}{\pi^{3}}G(\alpha,\beta,\gamma,\lambda,\mu)\,|\alpha,\beta,\gamma\rangle_{\lambda,\mu}\,, \tag{53}$$

where $G(\alpha,\beta,\gamma,\lambda,\mu)$ is the expansion coefficient

$$G(\alpha,\beta,\gamma,\lambda,\mu) = \cosh^{2}\mu_{\lambda,\mu}\langle\langle\alpha,\beta,\gamma|\,\rangle_{1234}\,. \tag{54}$$

If Annie is able to teleport the 4-mode nonlinear state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ to Johnsum, then she can teleport $|\,\rangle_{1234}$ since it can be expanded by the set of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$. Assuming Annie and Johnsum share a quantum channel composing of four bipartite nonlinear entangled states $|\eta_a\rangle_{5,6}\otimes|\eta_b\rangle_{7,8}\otimes|\eta_c\rangle_{9,10}\otimes|\eta_d\rangle_{11,12}$, which means that Annie owns particle 5, 7, and 9, 11, while Johnsum owns 6, 8, and 10, 12. Then initial state of the whole system is $|\Psi_{in}\rangle = |\alpha,\beta,\gamma\rangle_{\lambda,\mu 1234}\otimes|\eta_a\rangle_{5,6}\otimes|\eta_b\rangle_{7,8}\otimes|\eta_c\rangle_{9,10}\otimes|\eta_d\rangle_{11,12}$. Annie carries a joint Bell measurement, the projection state is $|\eta\rangle_{15}\otimes|\eta'\rangle_{27}\otimes|\eta''\rangle_{39}\otimes|\eta'''\rangle_{4,11}$, then using the Schmidt decomposition of $|\eta\rangle_{15}$,

$$_{15}\langle\langle\eta| = e^{\frac{i\eta_1\eta_2}{2}}\int_{-\infty}^{\infty}dx_1\,\langle\langle x|\otimes_5\langle\langle x-\eta_1|\,e^{-i\eta_2 x}, \tag{55}$$

and using (49) we can calculate

$$\begin{aligned}|\Psi_{out}\rangle &= {}_{4,11}\langle\langle\eta'''|\otimes_{39}\langle\langle\eta''|\otimes_{27}\langle\langle\eta'|\otimes_{15}\langle\langle\eta|\,\Psi_{in}\rangle\\ &= e^{i\Phi}\frac{1}{8\pi^{3}}\int_{-\infty}^{\infty}dudvd\sigma\, e^{-(iu\alpha_2+iv\beta+i\sigma\gamma_2)}W(u,v,\sigma)\\ &\quad\times e^{i((\eta_{a2}-\eta_2)x_1-\eta_{a2}\eta_2)}\,|x_1-\eta_1-\eta_{a1}\rangle_6\\ &\quad\otimes e^{i((\eta_{b2}-\eta'_2)x_2-\eta_{b2}\eta'_2)}\,|x_2-\eta'_1-\eta_{b1}\rangle_8\\ &\quad\otimes e^{i((\eta_{c2}-\eta'_2)x_3-\eta_{c2}\eta'_2)}\,|x_3-\eta''_1-\eta_{c1}\rangle_{10}\\ &\quad\otimes e^{i((\eta_{d2}-\eta'_2)x_4-\eta_{d2}\eta'_2)}\,|x_4-\eta'_1-\eta_{d1}\rangle_{12}\,,\end{aligned} \tag{56}$$

where $\Phi = (\eta_1\eta_2 + \eta_1'\eta_2' + \eta_1''\eta_2'' + \eta_1'''\eta_2''' - \eta_{a1}\eta_{a2} - \eta_{b1}\eta_{b2} - \eta_{c1}\eta_{c2} - \eta_{d1}\eta_{d2})/2$. Note that

$$\begin{aligned} &e^{i((\eta_{a2}-\eta_2)x_1-\eta_{a2}\eta_2)}\,|x_1-\eta_1-\eta_{a1}\rangle_6 \\ &\quad = e^{i((\eta_{a2}-\eta_2)x_6+\eta_{a1}\eta_{a2}-\eta_2(\eta_1+\eta_{a1}))}e^{ip_6(\eta_1+\eta_{a1})}\,|x_1\rangle_6\,, \end{aligned} \tag{57}$$

$$\begin{aligned} &e^{i\left((\eta_{b2}-\eta_2')x_2-\eta_{b2}\eta_2'\right)}\,\left|x_2-\eta_1'-\eta_{b1}\right\rangle_8 \\ &\quad = e^{i\left((\eta_{b2}-\eta_2')x_8+\eta_{b1}\eta_{b2}-\eta_2'(\eta_1'+\eta_{b1})\right)}e^{ip_8(\eta_1'+\eta_{b1})}\,|x_2\rangle_8\,, \end{aligned} \tag{58}$$

and

$$\begin{aligned} &e^{i\left((\eta_{c2}-\eta_2')x_3-\eta_{c2}\eta_2'\right)}\,\left|x_3-\eta_1'-\eta_{c1}\right\rangle_{10} \\ &\quad = e^{i\left((\eta_{c2}-\eta_2')x_{10}+\eta_{c1}\eta_{c2}-\eta_2'(\eta_1'+\eta_{c1})\right)}e^{ip_{10}(\eta_1'+\eta_{c1})}\,|x_3\rangle_{10}\,, \end{aligned} \tag{59}$$

$$\begin{aligned} &e^{i\left((\eta_{d2}-\eta_2')x_4-\eta_{d2}\eta_2'\right)}\,\left|x_4-\eta_1'-\eta_{d1}\right\rangle_{12} \\ &\quad = e^{i\left((\eta_{d2}-\eta_2')x_{12}+\eta_{d1}\eta_{d2}-\eta_2'(\eta_1'+\eta_{d1})\right)}e^{ip_{12}(\eta_1'+\eta_{d1})}\,|x_4\rangle_{12}\,. \end{aligned} \tag{60}$$

Substituting (57–60) into (56) we obtain

$$\begin{aligned} |\Psi_{out}\rangle &= e^{i\Phi'}U_xU_p\frac{1}{8\pi^3}\int_{-\infty}^{\infty}dud\upsilon d\sigma\, e^{-(iu\alpha_2+i\upsilon\beta+i\sigma\gamma_2)}W(u,\upsilon,\sigma) \\ &\quad \times|x_1\rangle_6\otimes|x_2\rangle_8\otimes|x_3\rangle_{10}\otimes|x_4\rangle_{12} \\ &= e^{i\Phi'}U_xU_p\,|\alpha,\beta,\gamma\rangle_{\lambda,\mu 6,8,10,12}\,, \end{aligned} \tag{61}$$

where

$$e^{i\Phi'} = e^{\frac{-i\left(\eta_1\eta_2+\eta_1'\eta_2'+\eta_1'\eta_2'+\eta_1'\eta_2'-\eta_{a1}\eta_{a2}-\eta_{b1}\eta_{b2}-\eta_{c1}\eta_{c2}-\eta_{d1}\eta_{d2}\right)}{2}}e^{-i\left(\eta_2\eta_{a1}+\eta_2'\eta_{b1}+\eta_2'\eta_{c1}+\eta_2'\eta_{d1}\right)}, \tag{62}$$

and

$$U_x = e^{i(\eta_{a2}-\eta_2)x_6}e^{i\left(\eta_{b2}-\eta_2'\right)x_8}e^{i\left(\eta_{c2}-\eta_2'\right)x_{10}}e^{i\left(\eta_{d2}-\eta_2'\right)x_{12}}, \tag{63}$$

$$U_p = e^{ip_6(\eta_1+\eta_{a1})}e^{ip_8(\eta_1'+\eta_{b1})}e^{ip_{10}(\eta_1'+\eta_{c1})}e^{ip_{12}(\eta_1'+\eta_{d1})}. \tag{64}$$

Then Annie informs Johnsum of the date of η, η', η'', and η''' via a classical channel, and Johnsum, up to an over phase factor, makes the unitary transform $(U_xU_p)^{-1}$, and successfully obtains the unknown state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$.

7 Conclusion

In summary, we have introduced a kind of four-mode nonlinear entangled state $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ and analyzed its Schmidt decomposition as well as a quadrature amplitude measurement on it. It is found that $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ possesses the properties of two- and three-mode entangled state. We have proved the completeness relation of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ and show that $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ is partly orthogonal. A possible scheme to generate this state has been presented. The set of $|\alpha,\beta,\gamma\rangle_{\lambda,\mu}$ is very important because it makes up a new quantum mechanical representation, which may enrich Dirac's representation theory in the context of multi-partite nonlinear entangled states.

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